

A conjecture on the infrared structure of the vacuum Schrödinger wave functional of QCD

John M. Cornwall*

Department of Physics and Astronomy, University of California, Los Angeles CA 90095

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The Schrödinger wave functional $\psi = \exp -S\{\mathcal{A}_i^a(\vec{x})\}$ for the $d = 3+1$ QCD vacuum is a partition function constructed in $d = 4$; the exponent $2S$ [in $|\psi|^2 = \exp(-2S)$] plays the role of a $d = 3$ Euclidean action. We start from a simple conjecture for S based on dynamical generation of a gluon mass M in $d = 4$, then use earlier techniques of the author to extend (in principle) the conjectured form to full non-Abelian gauge invariance. We argue that the exact leading term, of $\mathcal{O}(M)$, in an expansion of S in inverse powers of M is a $d = 3$ gauge-invariant mass term (gauged non-linear sigma model); the next leading term, of $\mathcal{O}(1/M)$, is a conventional Yang-Mills action. The $d = 3$ action that is (twice) the sum of these two terms has center vortices as classical solutions. The $d = 3$ gluon mass m_3 , which we constrain to be the same as M , and $d = 3$ coupling g_3^2 are related through the conjecture to the $d = 4$ coupling strength, but at the same time the dimensionless ratio m_3/g_3^2 can be estimated from $d = 3$ dynamics. This allows us to estimate the $d = 4$ coupling $\alpha_s(M^2)$ in terms of the strictly $d = 3$ ratio m_3/g_3^2 ; we find a value of about 0.4, in good agreement with an earlier theoretical value but somewhat low compared to the QCD phenomenological value of 0.7 ± 0.3 . The wave functional for $d = 2 + 1$ QCD has an exponent that is a $d = 2$ infrared-effective action having both the gauge-invariant mass term and the field strength squared term, and so differs from the conventional QCD action in two dimensions, which has no mass term. This conventional $d = 2$ QCD would lead in $d = 3$ to confinement of all color-group representations. But with the mass term (again leading to center vortices), only N -ality $\not\equiv 0 \pmod N$ representations can be confined (for gauge group $SU(N)$), as expected.

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I. INTRODUCTION

The functional Schrödinger equation (FSE) for gauge theories, while no simpler to solve (and perhaps harder, in some ways) than any other non-perturbative formulation of QCD, has often been used over the years to gain insight into various aspects of QCD or, more generally, $SU(N)$ gauge theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. However, few of these works address the important question of how confinement is expressed in the FSE.

In any approach to the FSE for QCD that purports to reveal confinement, there are two important prerequisites: The first is gauge invariance, and it has been addressed many ways. The second is the need to insure that there are only short-range field-strength correlations; otherwise (see, *e. g.*, the qualitative and in some ways incomplete discussion of Feynman [6]) there cannot be confinement. Given these, confinement further requires long-range pure-gauge contributions to the potential. These long-range pure-gauge parts appear in the FSE as massless longitudinally-coupled scalars that mimic Goldstone fields, although of course there is no symmetry breaking in QCD. Just as with conventional Goldstone fields, these massless poles do not appear in the QCD S-matrix; this would be so even if QCD were not a confining theory. As is well-known, center vortices, solitons of an infrared-effective action for QCD that encapsulates dynamical and gauge-invariant generation [18, 19] of a gluon mass M , show just these properties and so provide a confinement mechanism. This mass has been estimated theoretically [18], from phenomenology [20, 21], and on the lattice [22], all yielding values of 600 ± 200 MeV. The center vortices in $d = 3$ are characterized by closed strings that (generically) constitute the constant-time cross-sections of $d = 4$ center vortices; a confining condensate of center vortices in $d = 4$ is therefore mirrored by a similar condensate in $d = 3$. (Of course, the classical local minimum describing a single or a few center vortices is not relevant in isolation; it is necessary that there be an entropy-driven condensate of vortices. We do not discuss that issue here.)

In $d = 3 + 1$ the FSE describes four-dimensional dynamics in $d = 3$ terms, because the exponent S in the vacuum wave functional $\psi = \exp(-S)$ is (half of) a $d = 3$ action, which we label $I_{d=3}$, depending on the spatial gauge

* Email: Cornwall@physics.ucla.edu

potentials at zero time. Many authors have discussed center vortices for QCD in strictly $d = 4$ terms. Our question is, how are such solitons—and hence confinement—described in the FSE action $2S = I_{d=3}$?

Our answer proceeds in four steps. The FSE exponent S is an infinite series of n -point functions integrated over the spatial components of n gauge potentials (see the Appendix, which reviews earlier work [8] on the FSE, as well as Sec. III). The first step, described in Sec. II, considers the lowest-order term S_2 of this expansion, which is quadratic and shows only Abelian $U(1)^{N^2-1}$ gauge invariance. Our conjectured form of S_2 exactly satisfies the FSE with an Abelian gauge Hamiltonian that phenomenologically describes a gauge-invariant gluon mass M ; it is essentially $N^2 - 1$ copies of the Abelian Higgs model with infinite Higgs mass.

Since our focus is on confinement, an infrared phenomenon, we will use techniques and approximations that are useful in the infrared regime, even though they may misstate ultraviolet-dominated phenomena. In particular, although we treat the gluon mass M as a constant, it is actually a running mass $M(k^2)$ evaluated on-shell. In order that there be dynamical mass generation in QCD, the running mass must vanish for large momentum k^2 [18]. This vanishing cures certain short-distance singularities of the center-vortex solitons coming from an infrared-effective action. We will ignore this complication throughout this paper.

The Abelian case is not entirely trivial, since the action S_2 contains the square root of an operator—the hallmark of the FSE. (Throughout this paper, we take this operator, called Ω , in the simple form $\Omega = \sqrt{M^2 - \nabla^2}$.) Nonetheless, S_2 has center-vortex solutions. Although these do not completely coincide with conventional $d = 3$ center vortices, they show the necessary features: Long-range pure-gauge parts that confine, and field strengths that vanish at large distances as $\exp(-M\rho)$ where ρ is the distance from the closed string on which the vortex lives.

In anticipation of what we must do in the non-Abelian case, we study briefly the infrared expansion of S_2 in powers of k^2/M^2 , and show that the first two terms yield a familiar action. The leading term is a gauge-invariant mass term; the next-leading term is the usual Abelian gauge action. However, if the expansion is truncated after two terms, the gauge mass described by them is erroneous. The reason is elementary: The infrared expansion, at least in the Abelian case, is nothing but the first two terms of the expansion

$$\sqrt{M^2 - \nabla^2} \rightarrow \frac{1}{M}(M^2 - \frac{1}{2}\nabla^2 + \dots); \quad (1)$$

the two terms saved correspond to a mass $\sqrt{2}M$ instead of M . We propose that it may be phenomenologically useful, although not highly accurate, to make the replacement

$$\sqrt{M^2 - \nabla^2} \rightarrow \frac{Z}{M}(M^2 - \nabla^2) \quad (2)$$

where the renormalization constant $Z \simeq 1$ can be estimated in various ways. This heuristic replacement has the correct gluon mass. We discuss the motivation for this renormalization, coming from omitted terms in the infrared expansion.

The second step, the subject of Sec. III, begins with the problems of enforcing non-Abelian gauge invariance. Using earlier work [8], we show that S_2 of step one can be gauge-completed to exact non-Abelian gauge invariance with an infinite series of n -point functions and powers of gauge potentials, in such a way that all n -point functions depend only on the operator Ω , no matter what the specific form of Ω is. Gauge completion uses the pinch technique [18, 23] and the gauge technique [24], as reviewed in the Appendix. The gauge technique is an approximation that becomes exact only at zero momentum, but is useful generally for momenta not large compared to M . In this gauge-completed S we continue to use the simple form in S_2 of the two-point function introduced in the Abelian case. Just as for the ordinary Schrödinger equation, either direct substitution in the FSE or a dressed-loop expansion based on [25] ultimately yields a non-linear Schwinger-Dyson equation for Ω (see the Appendix). We do not attempt to carry out this difficult program to find Ω , but simply use the form already introduced in the Abelian case, showing mass generation.

Even the approximate (although showing full non-Abelian gauge invariance) form of S coming from application of the gauge/pinch technique is extremely complex, involving not only square roots of operators but an infinity of terms. This S looks nothing like actions that we are used to dealing with. Ultimately, whatever form S takes must be dealt with on its own terms. However, just as in the Abelian case it can be helpful to look for an approximate but familiar form. We make the same sort of mass expansion, saving only the first two terms, and argue that for QCD the leading term, of $\mathcal{O}(M)$, is equivalent to a gauged non-linear sigma (GNLS) model, which is commonly used as a description of gauge-invariant dynamical mass generation in Yang-Mills theory (see, for example, [18, 26]). This sigma model contains the massless scalar poles, actually pure-gauge parts of center vortices, that are responsible for confinement. The second term, of $\mathcal{O}(1/M)$, is (after gauge completion) the conventional Yang-Mills action. But as in the Abelian case, the mass is wrong by a factor $\sqrt{2}$, so we suggest using the replacement of Eq. (2).

In Sec. IV we give the final conjecture for the non-Abelian exponent S and $d = 3$ action $I_{d=3} \equiv 2S$, and the main consequences following from it. The conjectured action is the sum of a GNLS model and a conventional Yang-Mills

action, with the correct free-field mass and a poorly-known renormalization constant Z . We suggest a method or two for estimating Z , probably with no more than 25% accuracy.

The fourth step is to examine the consequences of this final two-term action. We have already noted that this action has center vortices as classical local minima (classical maxima of the FSE wave functional), and thus could provide a description of confinement in the FSE, which was one of our principal goals. Moreover, by appealing to known $d = 3$ gauge dynamics, we can estimate the $d = 4$ coupling strength in terms of the renormalization constant Z . In $d = 3$ the coupling g_3^2 has dimensions of mass, and there is a unique [for given $SU(N)$] dynamically-determined ratio M/g_3^2 , which has been estimated by a number of authors [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. Knowing only this ratio we can estimate the $d = 4$ QCD coupling $\alpha_s(M^2)$, getting a value around $0.4Z$. For $Z \simeq 1$ this is reasonably close both to an early $d = 4$ estimate [18] using the gauge technique and pinch technique but somewhat low compared to phenomenological estimates [21] of 0.7 ± 0.3 . Another application is to the $d = 2 + 1$ FSE, studied in, among other works, [2, 17]. Our present techniques suggest that the corresponding $d = 2$ FSE exponent S is again a sum of a gauge-invariant mass term and the usual Yang-Mills action. Greensite [2] speculated that this S just had the conventional Yang-Mills term. However, as noted there and in [17], this would lead to the wrong conclusion that in $d = 2 + 1$ all representations of $SU(N)$ were confined, when in fact the adjoint and other representations with N -ality $\equiv 0 \bmod N$ are screened, not confined. But with the addition of the mass term, confinement can come about through center vortices, and this form of confinement correctly predicts screening for these representations.

The paper ends with Sec. V, giving conclusions. An Appendix reviews some background material on the FSE, including applications of the pinch/gauge technique to the gauge FSE.

II. DESCRIBING MASS GENERATION IN THE FSE: THE ABELIAN CASE

Notation: Throughout this paper we will always use the canonical gauge potential $A_i^a(\vec{x})$ potential multiplied by the coupling g , with the notation:

$$A_i^a(\vec{x}) = g\mathcal{A}_i^a(\vec{x}). \quad (3)$$

Here a is a group index for gauge group $SU(N)$, and $i = 1, 2, 3$ index the spatial components. All vectors are three-dimensional, so we will now drop the vector notation and just use, *e.g.*, k for a three-momentum. We also use the antihermitean matrix form

$$A_i(x) = \left(\frac{g}{2i}\right)\lambda_a\mathcal{A}_i^a(x) \quad (4)$$

where the λ_a are the Gell-Mann matrices for $SU(N)$, obeying

$$Tr \frac{1}{2}\lambda_a \frac{1}{2}\lambda_b = \frac{1}{2}\delta_{ab}. \quad (5)$$

The A_i^a have engineering mass dimension 1 in any dimension. The time component A_i^0 is missing from the FSE. In this paper we will not need to indicate gauge-fixing and ghost terms necessary to define the $d = 3$ functional integrals that yield physical expectation values.

In the first step we begin with a simple quadratic (in the gauge potentials) form for S that is consistent with gluon mass generation. This quadratic form S_2 is Abelian, showing $U(1)^{N^2-1}$ local gauge invariance:

$$S_2 = \frac{1}{2g^2} \int A_i^a \Omega_{ij} A_j^a(x) \quad (6)$$

where the integral is over three-space, and Ω_{ij} is a product of two factors:

$$\Omega_{ij} = P_{ij}\Omega. \quad (7)$$

The factor P_{ij} is a transverse projector:

$$P_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \quad (8)$$

that is required for Abelian gauge invariance. The free-field value of Ω , called Ω_0 , describes free massless particles:

$$\Omega_0 = \sqrt{-\nabla^2} = \sqrt{k^2} \quad (9)$$

where k is a three-momentum. To describe dynamical mass generation we will use, in this paper, the simple form

$$\Omega = \sqrt{-\nabla^2 + M^2} \quad (10)$$

in which the gluon mass M is the on-shell value of a running mass. Putting these equations together we have:

$$S_2 = \frac{1}{2g^2} \int A_i^a \sqrt{M^2 - \nabla^2} P_{ij} A_j^a. \quad (11)$$

One can easily check that S_2 is an exact solution to the FSE for an Abelian Hamiltonian with a gauge-invariant mass term:

$$H = \int \left\{ -\frac{1}{2} g^2 \left(\frac{\delta}{\delta A_i^a} \right)^2 + \frac{1}{2g^2} \left[\frac{1}{2} (F_{ij}^a)^2 + M^2 A_i^a P_{ij} A_j^a \right] \right\} \equiv \int \left[\frac{1}{2} (\Pi_i^a)^2 \right] + V. \quad (12)$$

where $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a$ are the Abelian field strengths. Here the mass term is put in by hand; in the non-Abelian version, we imagine that this mass term summarizes the effects of non-Abelian condensates.

A. Equations of motion and solitons for S_2

One goal in this Abelian example is to find center vortex-like solitons as extrema of S_2 . It may not be entirely obvious how to proceed, because this action has the square root of an operator, leads to subtleties concerning positivity, locality, and self-adjointness. For example, we will see that the operator $\sqrt{M^2 - \nabla^2}$ effectively vanishes on center vortex solitons, although $-\nabla^2$ is formally positive; this would falsely suggest that the action of such a soliton is zero. Consider the following alternative description of S_2 , found by expanding the square root in powers of $-\nabla^2/M^2$ and assuming that integration by parts with no boundary terms is allowed at all orders:

$$S_2 = \frac{M}{2g^2} \int A_i^a P_{ij} A_j^a + \frac{1}{4g^2} \int \sum_{N=0} C_{N+1} M^{-1-2N} [\partial_1 \dots \partial_N F_{ij}^a(x)]^2 \quad (13)$$

where $\partial_k \equiv \partial/\partial x_k$ and the C_N are the coefficients of x^N in the power-series expansion of $\sqrt{1+x}$. This re-definition of the square root gives the same generalized Euler-Lagrange equations as the naive equations following from the original form of Eqs. (6,7,10), because these equations assume that integrating by parts gives no contributions (as would be appropriate for functions that fall off at least exponentially).

In order to study these generalized Euler-Lagrange equations, it is very helpful to have S_2 in a formally local form. We note that, term by term, all but the first term of this alternative form of S_2 are both local and manifestly gauge-invariant, and need no change. As for the first term, we replace (in a familiar way) the non-local part by scalar fields:

$$S_2 = \frac{M}{2g^2} \int [A_i^a - \partial_i \phi^a]^2 + \frac{1}{8Mg^2} \int [F_{ij}^a]^2 + \dots \quad (14)$$

Now keeping only a finite number of terms in the mass expansion of S_2 yields a local action, although of course the infinite sum may introduce non-localities.

Saving only the first two terms in the mass expansion of S_2 based on Eq. (13) should fail to satisfy the Abelian FSE based on the Hamiltonian of Eq. (12). It is instructive to work out this failure and its consequences. The FSE reads:

$$-\frac{g^2}{2} \int \left(\frac{\delta S_2}{\delta A_i^a} \right)^2 + \frac{g^2}{2} \int \frac{\delta^2 S_2}{\delta A_i^a \delta A_i^a} + H = E \quad (15)$$

where E is the vacuum energy. Since the second-derivative term on the left-hand side of this equation only contributes to E , we drop it and renormalize E to zero. The mass expansion of S_2 suggests that the remaining quadratic term in the FSE is in error at $\mathcal{O}(1/M^2)$. A simple calculation confirms this; Eq. (15) becomes:

$$-\frac{g^2}{2} \int \left(\frac{\delta S_2}{\delta A_i^a} \right)^2 + H + \frac{1}{4g^2} \int \frac{1}{2M^2} (\partial_j F_{ij}^a)^2 = 0. \quad (16)$$

At least qualitatively this error term in the FSE [last term on the left-hand side, of $\mathcal{O}(1/M^2)$, the same relative order as the $N = 1$ term in Eq. (13)] can be thought of as increasing the kinetic field-strength term $(F_{ij}^a)^2$ by a factor

involving a mean-square momentum of the type $\langle k^2 \rangle / M^2$; such an increase helps restore the balance between kinetic and mass terms in the expanded Hamiltonian which was disrupted by the usual infrared expansion of Eq. (14). Such a renormalization is not quantitatively trivial, since momenta relevant for solitons such as center vortices are of $\mathcal{O}(M)$.

It is useful to restate the local form of S_2 in a compact way, by undoing the power-series expansion and integration by parts:

$$S_2 = \frac{M}{2g^2} \int [A_i^a - \partial_i \phi^a]^2 + \frac{1}{2g^2} \int A_i^a P_{ij} [\sqrt{M^2 - \nabla^2} - M] A_j^a. \quad (17)$$

The scalar fields ϕ^a are to be integrated over, which may be thought of as projection of a simple mass term $(A_i^a)^2$ onto its gauge-invariant part by integrating over all gauge transformations. Because the ϕ^a appear quadratically, such an integration is the same as solving the classical field equations. The field equations for the ϕ^a are identical with a constraint following from the field equations for the A_i^a .

Varying S_2 , one finds the gauge potential equations of motion:

$$M(A_i^a - \partial_i \phi^a) + [\sqrt{M^2 - \nabla^2} - M] P_{ij} A_j^a = 0. \quad (18)$$

The divergence yields the ϕ^a equations:

$$\nabla^2 \phi^a = \partial_i A_i^a \rightarrow \phi^a = \frac{1}{\nabla^2} \partial_i A_i^a + \varphi^a \text{ with } \nabla^2 \varphi^a = 0. \quad (19)$$

Re-write Equation (19) as:

$$\sqrt{M^2 - \nabla^2} P_{ij} A_j^a = M \partial_i (\phi^a - \frac{1}{\nabla^2} \partial_j A_j^a) = M \partial_i \varphi^a. \quad (20)$$

Multiplication by $\sqrt{M^2 - \nabla^2}$ leads to:

$$\begin{aligned} (M^2 - \nabla^2) P_{ij} A_j^a &= M \sqrt{M^2 - \nabla^2} \partial_i \varphi^a \rightarrow \\ \nabla^2 A_i^a - \partial_i \partial_j A_j^a &= M^2 (A_i^a - \partial_i \frac{1}{\nabla^2} \partial_j A_j^a) - M \sqrt{M^2 - \nabla^2} \partial_i \varphi^a \rightarrow \\ \nabla^2 A_i^a - \partial_i \partial_j A_j^a - M^2 (A_i^a - \partial_i \phi^a) &= M [M - \sqrt{M^2 - \nabla^2}] \partial_i \varphi^a. \end{aligned} \quad (21)$$

Term by term, every term on the right-hand side of the third equation in Eq. (21) vanishes, if we use $\nabla^2 \varphi^a = 0$. Since in \mathcal{R}^3 there are no fields φ^a solving $\nabla^2 \varphi^a = 0$ that are regular everywhere and vanish at infinity, one may be tempted to make the stronger statement that φ^a must vanish. But the description of center vortices requires a non-zero φ^a , singular on a closed Dirac hypersurface of co-dimension 2 (a closed string in $d = 3$), so it is more accurate to say that term by term the expansion of the right-hand side of the third equation in Eq. (21) vanishes almost everywhere. However, we will soon see that this is not true for the unexpanded form. If we nonetheless drop this term with the square-root operator, the final equations of motion are the usual equations [38] for center vortices, the same as would be gotten from the $d = 3$ Euclidean action

$$\frac{1}{2g^2} \int \{ M^2 (A_i^a - \partial_i \phi^a)^2 + \frac{1}{2} (F_{ij}^a)^2 \}. \quad (22)$$

This action is just the potential V occurring in the Abelian Hamiltonian of Eq. (12), written in local form; it is the Abelian form of the $d = 3$ infrared-effective action used [18, 38] to describe mass generation, and it has center vortices as classical solitons.

If the term $M[M - \sqrt{M^2 - \nabla^2}] \partial_i \varphi^a$ is left unexpanded, things are slightly different, although there are still center vortices characterized by long-range pure-gauge parts and field strengths vanishing exponentially as $\exp(-M\rho)$, where ρ is the distance from the Dirac string. A center vortex is always fully determined by φ^a . We present our results in the gauge $\partial_i A_i^a = 0$, in which case $\phi^a = \varphi^a$. The well-known expression [38] for the center-vortex $\partial_i \phi^a$ is:

$$\partial_i \phi^a(x) = 2\pi Q^a \epsilon_{ijk} \partial_j \oint_{\Gamma} dz_k \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2} e^{ik \cdot (x-z)} \quad (23)$$

where the closed contour Γ is the Dirac string, and Q^a is one of the $N - 1$ generators of the Cartan subalgebra, normalized so that $\exp(2\pi i Q)$ is in the center of $SU(N)$. Now the third equation in Eq. (21) easily gives:

$$A_i^a(x) = 2\pi Q^a \epsilon_{ijk} \partial_j \oint_{\Gamma} dx_k \int \frac{d^3 k}{(2\pi)^3} \frac{M}{k^2 \sqrt{k^2 + M^2}} e^{ik \cdot (x-z)}. \quad (24)$$

In the usual $d = 3$ vortex, an extremum of the action in Eq. (22), the factor $M(k^2 + M^2)^{-1/2}$ would be replaced by $M^2(k^2 + M^2)^{-1}$.

This unusual square root does not change the fact that the field strengths show exponential decrease; in fact:

$$B_i^a = \frac{1}{2}\epsilon_{ijk}F_{jk}^a = 2\pi Q^a \oint_{\Gamma} dz_i \frac{M^2}{2\pi^2|x-z|} K_1(M|x-z|). \quad (25)$$

There is, of course, still the long-range pure-gauge part associated with ϕ^a , which we can isolate by the decomposition:

$$\frac{M}{k^2\sqrt{k^2 + M^2}} = \frac{1}{k^2} + \frac{1}{k^2} \left\{ \frac{M}{\sqrt{k^2 + M^2}} - 1 \right\}. \quad (26)$$

The second term on the right-hand side is short-ranged. The short distance behavior is more singular than that of the conventional vortex, but leads only to a logarithmic singularity in the value of S_2 , the same as for the conventional vortex. In both cases the singularity is multiplied by a power of M , which removes the singularity because the running mass vanishes at short distances. So the vortex extrema of S_2 differ in detail from the usual center vortex, but have the hallmark features of a long-range pure-gauge part and field strengths vanishing like $\exp(-M\rho)$.

B. Mass expansion of S_2

Another goal of this section is to replace S_2 , which is either given in Eq. (13) as an infinite sum involving derivatives of arbitrarily high order or in Eq. (17) in terms of square roots of operators, by a tractable and recognizable action. The first two terms of the expansion, written explicitly in Eq. (14), fit these criteria, but suffer from a serious defect. The coefficient of the second term, the usual gauge action, is wrong by a factor of 2; as written, it describes gauge bosons of mass $\sqrt{2}M$. This wrong coefficient arises from the expansion $\sqrt{1+x} = 1 + (x/2) + \dots$. We can see the same thing happening with a mass expansion of the Fourier kernel of Eq. (26). Expand the square root in the curly brackets of this equation in powers of k^2/M^2 to get:

$$\frac{M}{k^2\sqrt{k^2 + M^2}} = \frac{1}{k^2} - \frac{1}{k^2 + 2M^2} + \dots \quad (27)$$

This is exactly the kernel of the usual $d = 3$ vortex, but with the wrong mass $\sqrt{2}M$. This is not the only way of expanding; for example, re-writing the Fourier kernel in a different form and expanding the square root occurring in it gives:

$$\frac{M}{k^2\sqrt{k^2 + M^2}} = \frac{M\sqrt{k^2 + M^2}}{k^2(k^2 + M^2)} = \frac{M^2}{k^2(k^2 + M^2)} - \frac{1}{2(k^2 + M^2)} + \dots \quad (28)$$

The first term on the right-hand side is the standard center vortex with the correct mass M , and all other terms have this mass as well. However, these other terms give the wrong coefficient for the exponential falloff of the field strengths at large distance.

There are no such results for square-root operators in the non-Abelian case, which is as expected much more complicated. So we will, in the spirit of the Abelian expansion given in Eq. (14), look for a way to approximate the complicated non-Abelian result by a two-term form, the first of which is a (gauge-invariant) mass term and the second is the usual Yang-Mills action. In the Abelian case, such a two-term action as an approximation to the infinite sum of Eq. (13) suggests that the derivatives in this sum, beyond those in F_{ij}^a , be approximated by averages so that this equation is effectively

$$S_2 = \frac{M}{2g^2} \int A_i^a P_{ij} A_j^a + \frac{1}{4Mg^2} \int \sum_{N=0} C_{N+1} \langle \frac{k^{2N}}{M^{2N}} \rangle [F_{ij}^a(x)]^2 \equiv \frac{M}{2g^2} \int A_i^a P_{ij} A_j^a + \frac{Z}{4Mg^2} \int [F_{ij}^a(x)]^2 \quad (29)$$

where k^{2N} stands for the multiple derivatives. If this is justified, the infinitely many terms of Eq. (13) are indeed replaceable by a mass term plus a renormalized conventional gauge action. But because the gluonic mass described by this S_2 must be M , the same as in the original S_2 , there will have to be an equal renormalization of the mass term in Eq. (29) above. In later sections we will explore an approximation to the square root that is motivated by these remarks, involving the replacement

$$\sqrt{M^2 - \nabla^2} \rightarrow \frac{Z}{M}(M^2 - \nabla^2) \quad (30)$$

for some renormalization constant Z , supposed to be near unity. We have no reliable techniques for calculating Z , so we will resort to a simplistic approach of making a least-squares fit of the operator $\sqrt{M^2 - \nabla^2}$ by the operator $(Z/M)(M^2 - \nabla^2)$, which leads to $Z \simeq 1.1 - 1.2$.

Before engaging in this mass expansion we must understand the gauge structure of the non-Abelian exponent S .

III. THE NON-ABELIAN CASE: GAUGE COMPLETION AND MASS EXPANSION

We can be nowhere near as complete in the non-Abelian case as we were above, and ultimately will be forced to resort to a large-mass expansion.

In the non-Abelian case, the quadratic term S_2 with which we began is supplemented with an infinity of terms, involving spatial integrals over $n \geq 3$ spatial gauge potentials multiplied by an n -point function Ω_n depending on the spatial and discrete coordinates of the gauge potentials (see the Appendix):

$$g^2 S = \frac{1}{2!} \int \int A_i^a \Omega_{ij} A_j^a + \frac{1}{3!} \int \int \int A_i^a A_j^b A_k^c \Omega_{ijk}^{abc} + \dots \quad (31)$$

The n -point function of this expansion is related to the $n + 1$ -point function through ghost-free Ward identities, as arise in the pinch technique [18, 23]. These Ward identities can be “solved” using the gauge technique, a well-known technique whose main points of interest we describe in the Appendix, and the result is that it is possible in principle to find an approximate but exactly gauge-invariant expression for the entire series of n -point functions describing the wave functional exponent S . Each n -point function depends only on the two-point function, but in a complicated way that is not understood. Ultimately the two-point function is determined by a non-linear Schwinger-Dyson equation that can (again in principle) be derived either by direct substitution in the FSE or by a dressed-loop expansion [25, 39, 40]. Using a dressed-loop expansion for S is equivalent to direct solution of the FSE (of course, either the dressed-loop expansion or the FSE must be truncated at a certain number of loops, but this truncation has nothing to do with a truncation in the coupling g^2 ; all-order non-perturbative effects arise even at one-dressed-loop order in QCD).

A systematic study of the FSE would go on to determine the mass M from the infinity of equations for the n -point functions in S of Eq. (31), but that is not our purpose here. Instead, we show how to construct what we will call a *gauge completion* of the 2-point action for an arbitrary Ω , using earlier work [8], to add higher-point functions, consistent with solving the FSE, that depend on Ω in specific ways that insure full non-Abelian gauge invariance. Ultimately, the FSE becomes a non-linear equation for Ω , just as for the ordinary Schrödinger equation.

Gauge invariance requires that the lowest-order (quadratic) term has the Abelian form already given in Eq. (11). The Ward identities for the three-point function and their solution are detailed in the Appendix. Both the Ward identities and the FSE for the determination of this three-point function involve only the two-point function Ω_{ij} , and it is plausible that there exists a three-point function satisfying these equations that is a functional solely of the two-point function Ω_{ij} . The gauge technique provides such a three-point function, as given in Eqs. (A20, A21). The gauge technique by itself does not furnish a unique solution, which must be found by recourse either to the FSE itself or to the dressed-loop expansion. However, in the infrared limit of momenta small compared to the mass M the solution is unique.

A. Mass expansion: The leading term

In general, the gauge/pinch technique leads to quite complicated expressions, and we will explore only a simplified version of it. The main simplification is to look at the leading terms in an expansion in inverse powers of M . In the leading term, of $\mathcal{O}(M)$, all two-point functions Ω are replaced just by M itself, which gets rid of many momentum-dependent terms. In this way the leading term of the three-point function is:

$$\Omega_{ijk}^{abc}(k_1, k_2, k_3) = f^{abc} \frac{M}{6} \left\{ \frac{k_{1i} k_{2j} (k_1 - k_2)_k}{k_1^2 k_2^2} + c.p. \right\} + \mathcal{O}(1/M). \quad (32)$$

One can proceed in principle this way, by looking at the pinch/gauge technique solution for the four-point function (see [41]) and taking the large-mass limit, then the five-point function, etc. We will not detail such an investigation here, but will point out some features that strongly suggest the all-order solution. The structure of the Ward identities shows that the leading term of every n -point function is $\mathcal{O}(M)$, with all other dimensions taken up by momenta, and that the gauge-technique solution involves longitudinally-coupled massless poles whose number grows with n . Observe further that the GNLS term of S is the exact solution of an FSE Hamiltonian consisting of just this term itself, as

given in Eq. (33) below, multiplied by M . Of course, there is no such term in the underlying QCD Hamiltonian, but there would be one in the infrared-effective Hamiltonian of QCD, derived by $d = 4$ techniques [18, 26].

We suggest that the action of the gauged non-linear GNLS model, expressed in non-local form [as in the originally-stated form of S_2 , in Eq. (11)] is the all-order perturbative solution to the leading mass terms of the gauge/pinch technique approach. To find this non-local form we investigate the classical solutions of the local GNLS action.

Because the notation is more compact, we temporarily switch to the antihermitean matrix notation of Eq. (4). The local GNLS model, normalized appropriately, has the action [26]:

$$I_{GNLS} = \frac{-M}{g^2} \int d^3x \text{Tr}[U^{-1} D_i U]^2 \quad (33)$$

where U is a unitary matrix transforming as $U \rightarrow VU$ under the gauge transformation

$$A_i \rightarrow V A_i V^{-1} + V \partial_i V^{-1}. \quad (34)$$

The classical equations for U express this quantity in terms of the A_i [26], with the result

$$U = e^\omega; \quad \omega = \frac{-1}{\nabla^2} \partial \cdot A + \frac{1}{\nabla^2} \left\{ [A_i, \partial_i \frac{1}{\nabla^2} \partial \cdot A] + \frac{1}{2} [\partial \cdot A, \frac{1}{\nabla^2} \partial \cdot A] + \dots \right\} \quad (35)$$

showing the appearance of massless scalars. More generally, since $U^{-1} D_i U$ is a gauge transformation of A_i , functional integration over the U is equivalent to projecting out the gauge-invariant part of the mass term [8, 11]. Note that the term linear in A_i of the GNLS model field $U^{-1} D_i U$ is the transverse part of A_i . This linear term is Abelian, and all higher-order terms of ω in Eq. (35) are non-Abelian.

[Greensite and Olejnik [17] have conjectured that in certain instances operators such as ∇^{-2} should be replaced by D^{-2} , where $D_i = \partial_i + A_i$ is the covariant derivative. Their lattice calculations show that D^{-2} is a finite-range operator, with no massless poles; this is reasonable, because it contains gauge-potential condensate terms, but it is not obvious where the long-range pure-gauge excitations responsible for confinement, such as we have in Eq. (35), are. We will not follow this line of reasoning here.]

It is now straightforward, if tedious, to verify that the two- and three-point terms of the non-local GNLS action give rise precisely to (the leading mass terms of) the two-point function S_2 and the pinch/gauge technique three-point function of Eq. (32). Moreover, the GNLS action integrated over U automatically satisfies the Ward identities to all orders, just because it is the solution of the FSE for a gauge-invariant Hamiltonian.

B. The second-leading term

We already know that the next-leading term, of $\mathcal{O}(1/M)$, in the expansion of the two-point function S_2 is the conventional Abelian action involving F_{ij}^2 . It is obvious without any calculation that the Abelian action will, at a minimum, be gauge-completed to the full Yang-Mills action with its three- and four-point vertices. These come from the three- and four-point functions in the expansion of S as given in Eq. (31). The desired terms of the Yang-Mills action are straightforwardly found either by direct solution of the FSE or from the dressed-loop expansion, which always contain all the terms of the action of the underlying theory divided by some sum of two-point functions Ω . For example, we show in the Appendix that the three-point function has the term

$$\Omega_{ijk}^{abc}(k_1, k_2, k_3) = [\Omega(1) + \Omega(2) + \Omega(3)]^{-1} f^{abc} [\delta_{ij}(k_1 - k_2)_k + c.p.] + \dots \quad (36)$$

where the term in square brackets is the free Yang-Mills three-point vertex and each $\Omega(i)$ is replaced by M to find the leading term in the mass expansion. There is a plethora of other terms, which either cancel among themselves or give total divergences. Of course, higher-order gauge-invariant terms may arise from higher-order coefficient functions in the gauge-potential expansion of S , Eq. (A14), but we will not consider them, since they are necessarily accompanied by higher powers of $1/M$.

In the Abelian case the $\mathcal{O}(1/M)$ term is of the correct functional form, but with a coefficient twice as small as it should be, and the same problem arises for the non-Abelian case. This results in a gauge mass of $\sqrt{2}M$ instead of M , as pointed out in Sec. II. In the next section we consider a modification of the straightforward mass expansion of the type of Eq. (30) that forces the correct mass.

IV. THE FINAL CONJECTURE AND ITS CONSEQUENCES

A. Heuristic mass expansion

What we have so far in the gauge-completed mass expansion to second order is the sum of a GNLS and a Yang-Mills term, but with the wrong mass. What we need is an approximation to this two-term action that has the correct mass, in part because solitons are described in this momentum range and decay at a rate $\sim \exp(-M\rho)$. In any event, it is clear that the first two terms in any sensible infrared expansion consist first of a gauge-invariant mass term and second of a standard Yang-Mills action.

Rather than stick to a strict expansion in powers of ∇^2/M^2 , we conjecture that, as in the Abelian case, we can replace $\Omega = \sqrt{-\nabla^2 + M^2}$ by a leading term $(Z/M)(-\nabla^2 + M^2)$, where Z is a coefficient of $\mathcal{O}(1)$.

The mathematical motivation for least-square fits of operators is well-known. Consider a normal operator P , expressed in terms of its eigenvalues and eigenfunctions:

$$P = \sum |n\rangle \lambda_n \langle n|. \quad (37)$$

Any function of P , call it $f(P)$, is expressed by replacing λ_n by $f(\lambda_n)$. With the operator norm $\text{Tr} P^\dagger P$, we define a relative RMS distance between two operators $f(P)$ and $g(P)$ by:

$$\left\{ \frac{\text{Tr}[f(P) - g(P)][f(P) - g(P)]^\dagger}{\text{Tr} f(P)f(P)^\dagger} \right\}^{1/2} = \left\{ \frac{\int d\lambda \rho(\lambda) |f(\lambda) - g(\lambda)|^2}{\int d\lambda \rho(\lambda) |f(\lambda)|^2} \right\}^{1/2}, \quad (38)$$

where

$$\rho(\lambda) = \sum \delta(\lambda - \lambda_n) \quad (39)$$

is the density of eigenvalues. One could also modify this density by multiplying it by a non-negative function $q(\lambda)$ to emphasize a certain range of eigenvalues, so that the weight in the integral is $\rho(\lambda)q(\lambda)$.

The eigenvalues of $P = -\nabla^2$ are the squared momenta k^2 , positive for real k . We really want our approximation of Ω to be fairly good for imaginary k , so the above discussion is not very useful. Moreover, the operators involved are not in trace class, so divergences arise. Instead, we take a rather simpleminded point of view, asking what is the best fit, in the least-squares sense, of the function $Z(1 - x^2)$ to the function $\sqrt{1 - x^2}$ in the interval $0 \leq x \leq 1$. Here x^2 represents ∇^2/M^2 , and positive values for this operator suggest that we are applying it to a special class of functions representable by Laplace transformation, with the Laplace-transformation weight peaked around M . This is indeed the property of the functions that enter into FSE center vortices, as exemplified in the Abelian center vortex of Eq. (25).

For a uniform weight over $0 \leq x \leq 1$ we find the normalized least-squares integral $I_{ls}(Z)$:

$$I_{ls}(Z) = \left[\frac{\int_0^1 dx [Z(1 - x^2) - \sqrt{1 - x^2}]^2}{\int_0^1 dx (1 - x^2)} \right]^{1/2}. \quad (40)$$

Minimizing on Z gives $Z = \frac{45\pi}{128} \simeq 1.10$, and the minimum value of I_{ls} is about 0.22. If we replace x^2 by x in the integrand of Eq. (40), which corresponds to a different weight, we get $Z = 1.2$. Both values are near unity, as expected, and the value $I_{ls} \simeq 0.22$ suggests the relative accuracy of this least-squares fit.

B. The final conjecture: Relating d - and $d - 1$ -dimensional dynamics

The final form of the conjecture, expressed in terms of the $d = 4$ variables g^2, M is then:

$$-2S = -I_{d=3} \rightarrow \frac{2MZ}{g^2} \int d^3x \text{Tr}[U^{-1} D_i U]^2 + \frac{Z}{Mg^2} \int d^3x \text{Tr} G_{ij}^2 + \mathcal{O}(M^{-3}). \quad (41)$$

We now compare this to the canonical $d = 3$ form of the conjectured action, action, which is:

$$I_{d=3} = - \int d^3x \left\{ \frac{1}{2g_3^2} \text{Tr} G_{ij}^2 + \frac{m_3^2}{g_3^2} \text{Tr}[U^{-1} D_i U]^2 \right\}. \quad (42)$$

Here G_{ij} is the non-Abelian field strength, $D_i = \partial_i + A_i$ is the covariant derivative, and the unitary matrix U is the GNLS field, as before; the gluon mass is m_3 and the $d = 3$ coupling, with dimensions of mass, is g_3^2 . Equating $I_{d=3}$ with $2S$ leads to:

$$\frac{ZM}{g^2} = \frac{m_3^2}{2g_3^2}; \quad g^2 = \frac{2Zg_3^2}{M}. \quad (43)$$

These equations yield $m_3 = M$, as expected, plus

$$g^2 = \frac{2Zg_3^2}{m_3}. \quad (44)$$

(Note that the $d = 3$ quantities scale properly at large N if their $d = 4$ counterparts do.) Presumably the $d = 4$ coupling g^2 that occurs in these formulas is actually the running coupling $g^2(M^2)$ evaluated at the gluon mass scale.

We can now make an estimate of a pure $d = 4$ quantity in terms of a pure $d = 3$ quantity, from Eq. (44) and earlier $d = 3$ results. In $d = 3$ one quantity of particular interest is the dimensionless ratio m_3/g_3^2 . This ratio has been estimated in a number of continuum and lattice studies [27, 28, 29, 30, 31, 32, 33, 34, 35], and we can see whether this $d = 3$ dynamical quantity can correctly predict the running coupling $g^2(M^2)$ at the gluon mass scale. Or we can reverse the problem and use estimates of the running coupling to predict m_3/g_3^2 . There is no particular reason to think that the dynamics of the action defined by the exact vacuum wave functional, before truncation to two terms of a mass expansion, should be precisely that of $d = 3$ QCD. Nonetheless, if our conjecture is to be believed there should not be gross discrepancies.

In $SU(2)$ gauge theory various authors [27, 28, 29, 30, 31, 32, 33, 34, 35] give a value $m_3/g_3^2 \simeq 0.32$, and one $SU(3)$ lattice study [36] gives a value of 0.48. The quantity m_3/g_3^2 should be linear in N of $SU(N)$ for large N , and the factor $3/2$ nicely converts the $SU(2)$ values to the $SU(3)$ value, so we use 0.48 as the $SU(3)$ value. We then find a value for the strong coupling (with no quarks) $\alpha_s(M^2) \simeq 0.33Z$ that is in fairly good agreement with the one-dressed-loop approximation found in the original paper on dynamical gluon mass generation [18]. This paper gives a one-dressed-loop equation for the running charge with dynamical gluon mass generation. At the momentum scale of the gluon mass M :

$$\alpha_s(M^2) = \frac{g^2}{4\pi} = \frac{12\pi}{[11N - 2N_f] \ln[5M^2/\Lambda^2]} \simeq 0.4, \quad (45)$$

where the numerical value is based on the estimates $M = 0.6$ GeV, $\Lambda = 0.3$ GeV, and the absence of quarks ($N_f = 0$). Of course, these numbers for M and Λ are themselves uncertain, if only because Eq. (45) is a one-dressed-loop equation.

According to this one-dressed-loop equation, accounting for three light flavors multiplies the no-quark value by $11/9 \simeq 1.2$. If we assume that this correction applies to the FSE result of this paper, which as it stands does not account for quarks, our estimate of $\alpha_s(M^2)$ increases to about $0.4Z$.

Several papers have extracted values of $\alpha_s(0) \simeq 0.7 \pm 0.3$ from various scattering data sensitive to low-momentum effects [21] that could diverge if there were no gluon mass. The three-quark value that we give of $0.4Z$ is a little smaller, but in quite reasonable agreement considering the approximation that is inherent in a two-term truncation of the FSE exponent S and our lack of knowledge of Z .

It has been argued [37] that m_3/g_3^2 for $SU(N)$ is very closely approximated by the simple analytic function

$$\frac{m_3}{g_3^2} = \frac{N}{2\pi}; \quad (46)$$

the present author [18] has argued for a ratio that should be fairly close to $15N/(32\pi)$, which differs from the above by only a few percent. One then has a simple analytic formula for $\alpha_s(0)$. Using the value from Eq. (46) in Eq. (44) yields the amusing, if not very accurate, formula

$$\alpha_s(M^2) = \frac{Z}{N} \simeq \frac{1}{N}. \quad (47)$$

We can play the same game in one less dimension for the $d = 2 + 1$ FSE, beginning with an exponent S for the wave functional that is the trivial dimensional reduction of what we began with in $d = 3 + 1$. The result is a $d = 2$ action with, as in $d = 3 + 1$, a mass term and a kinetic term. This is not the standard $d = 2$ QCD action, which is a free field theory. We compare this to a conjecture made long ago by Greensite [2], arguing that S for the FSE was just the usual Yang-Mills action in one less dimension. Unfortunately, as [17] notes, if Greensite's 1979 conjecture is

applied in $d = 2 + 1$, the effective action is the familiar $d = 2$ free-field QCD, which would lead to confinement of all representations of $SU(N)$, not just those with N -ality nonzero. This is not the right behavior for $d = 2 + 1$. But in our case once again the action is the Yang-Mills term plus a GNLS model mass term; this action has [39] center vortices; they are point-like objects in $d = 2$. A condensate of these solitons leads to confinement, but only of group representations that have N -ality $\not\equiv 0 \bmod N$; other representations (such as the adjoint) are blind to the long-range parts of center vortices. This is the correct behavior for $d = 2 + 1$ gauge theories. However, if the mass term were not present in $I_{d=2}$ this action, which is supposed to carry all the information about $d = 2 + 1$ gauge theories, would reduce to the standard Yang-Mills action in $d = 2$. The conventional treatment of $d = 2$ gauge theories, which (in the absence of dynamical matter fields) are free field theories, finds confinement through the long-range free gluon propagator, and all representations are confined. But with the mass term the gluon propagator is short-ranged and confinement comes from the pure gauge long-range parts of center vortices.

It is far from trivial to calculate the properties of the center-vortex condensate in $d = 2$, and so we cannot relate the $d = 3$ coupling to the string tension that would be found from the $d = 2$ effective action.

V. SUMMARY AND CONCLUSIONS

We have conjectured that to a reasonable approximation the dominant quasi-infrared part of the vacuum wave functionals for the $d = 3 + 1$ and $d = 2 + 1$ FSE are actions in one less dimension consisting of a Yang-Mills term and a GNLS model term, showing gauge-invariant dynamical mass generation. Two main conclusions follow:

1. Given the usual entropy-dominance argument, these wave functionals show confinement through center vortices, such that only group representations with N -ality $\not\equiv 0 \bmod N$ are confined.
2. In $d = 3 + 1$ we can appeal to earlier works estimating the ratio m_3/g_3^2 in the $d = 3$ action of the FSE to make the estimate $\alpha_s(M^2) \simeq 0.4Z$, where Z is a renormalization constant that we have very crudely estimated to be in the neighborhood of 1.1-1.2. This can be compared to an earlier estimate, based on the original work on dynamical gluon mass generation, of $\alpha_s(M^2) \simeq 0.4$. Both these estimates have three light flavors of quarks. This is to be compared to phenomenological estimates [21], also with three light quarks, of $\alpha_s(0) \simeq 0.7 \pm 0.3$.

It would be interesting to verify this structure of the FSE vacuum wave functionals through lattice simulations.

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I am happy to acknowledge valuable conversations with Štefan Oleník about Ref. [17].

APPENDIX A: A BRIEF REVIEW OF THE FSE AND SOLUTION METHODS

The purpose of this review of known material [8] is to indicate the plausibility of constructing an infrared-accurate and gauge-invariant form of the wave functional ψ , based on a single operator Ω , obeying a non-linear Schwinger-Dyson equation. In ordinary quantum mechanics this is exactly what happens except that the ‘‘Schwinger-Dyson equation’’ is simply algebraic. The FSE for scalar field theories is really nothing but ordinary quantum mechanics for infinitely-many coupled oscillators, so we review it and its connection to quantum mechanics, then go on to gauge theories.

1. The Schrödinger equation

The general principles of solving the FSE in terms of an operator Ω are most easily understood from the ordinary Schrödinger equation. Consider the quadratic/quartic Hamiltonian

$$H = \frac{-1}{2} \left(\frac{d}{dx} \right)^2 + \frac{1}{2} \omega^2 x^2 + \frac{1}{4!} \lambda x^4. \quad (\text{A1})$$

The ground-state solution is $\psi = \exp(-S)$, with

$$S = \frac{1}{2} \Omega x^2 + \frac{1}{4!} \Omega_4 x^4 + \dots \quad (\text{A2})$$

Following [8], we substitute ψ in the Schrödinger equation saving only terms through Ω_6 and find:

$$\Omega_6 = \frac{-5\Omega_4^2}{3\Omega}; \quad \Omega_4 = \frac{\lambda}{4\Omega} + \frac{\Omega_6}{8\Omega}; \quad \Omega^2 = \omega^2 + \frac{1}{4}\Omega_4; \quad E = \frac{1}{2}\Omega. \quad (\text{A3})$$

It is easy to solve the equation for Ω_4 to derive a quartic equation for Ω . One can go on to any order this way, expressing (in principle, at least) every n -point coefficient up to a given highest value of n in terms of Ω , and ending up with a non-linear dressed-loop equation for Ω . Consider now the case $\omega = 0$, for which the perturbative expansion coefficient λ/ω^3 diverges. Then through the six-point term we find the expression $E = (3\lambda/272)^{1/3}$, which has the value $0.2226\lambda^{1/3}$. This is within a few percent of the numerical answer of $0.2311\lambda^{1/3}$.

2. Field theories other than gauge theories

For simplicity of exposition, we begin with a scalar field theory. Take the FSE Hamiltonian to be

$$H = \int d^3x \left[\frac{-1}{2} \left(\frac{\delta}{\delta\phi} \right)^2 + \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2 + V(\phi) \right]. \quad (\text{A4})$$

where the potential V contains cubic and higher terms. Ref. [8] showed that the vacuum wave functional could be expressed as a $d = 4$ partition function:

$$e^{-S} = \text{const.} \times \int (d\Phi) \exp[-I_0(\Phi) - I_0(\hat{\phi}_0) - \int V(\Phi + \hat{\phi}_0)]. \quad (\text{A5})$$

In this partition function, space-time integrals are of the form of an integral over a Euclidean time τ and all of three space:

$$\int_0^\infty d\tau \int d^3x. \quad (\text{A6})$$

The argument of S is the field $\phi(x)$, and the field $\hat{\phi}(x)$ depends on $x = (\tau, x)$ as:

$$\hat{\phi}_0(x) = e^{-\Omega_0 \tau} \phi(x) \quad (\text{A7})$$

with

$$\Omega_0 = \sqrt{M_0^2 - \nabla^2}. \quad (\text{A8})$$

The free action I_0 is:

$$I_0(\Phi) = \frac{1}{2} \int [(\partial_\tau \Phi)^2 + (\nabla \Phi)^2] \quad (\text{A9})$$

and the inverse of the free-action operator is the free propagator

$$\Delta_0 = \langle x | \frac{1}{2\Omega_0} [e^{-\Omega_0|\tau-\tau'|} - e^{-\Omega_0(\tau+\tau')}] | x' \rangle. \quad (\text{A10})$$

The first term in the propagator is the usual Euclidean propagator:

$$\langle x | \frac{1}{2\Omega_0} e^{-\Omega_0|\tau-\tau'|} | x' \rangle = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{ik \cdot (x-x')}}{k^2 + m^2}. \quad (\text{A11})$$

For purposes of calculating the energy eigenvalue, this is the only term that needs to be saved in Δ_0 , but the second term of the propagator in Eq. (A10) is necessary for calculating the wave functional.

Either by working out the partition function of Eq. (A5) or by direct substitution in the Schrödinger equation one sees that the vacuum functional ψ has the general form (using a streamlined but transparent notation):

$$\psi = e^{-S}; \quad S = \frac{1}{2} \int \int \phi \Omega \phi + \sum_N \frac{1}{N!} \int \dots \int \Omega_N \phi_1 \dots \phi_N. \quad (\text{A12})$$

For purely three-dimensional equations, such as this, the unadorned integral sign simply indicates $\int d^3x$, where x is the argument of a corresponding ϕ (and a sum over discrete indices, if any), with Ω and the $\Omega_N, N \geq 3$, as translationally-invariant form factors in the arguments of the ϕ . The partition function form in Eq. (A5) can be addressed with the well-known resummation techniques [25] of the dressed-loop expansion. The effect of these rules is to remove a large fraction of one-particle-reducible graphs, as required for the dressed-loop expansion. In part, this amounts to a general replacement (but not quite everywhere) of the free operator Ω_0 by a dressed operator Ω that satisfies a non-linear Schwinger-Dyson equation. This operator is precisely the same as the Ω that occurs in the quadratic term of the wave functional in Eq. (A12).

For further details of this formalism for scalar field theories, see [40] which uses it for calculating some terms in the Wigner distribution function.

3. Gauge theories

For gauge theories the same general structure holds; the principal problem remaining is to enforce gauge invariance. The canonical momentum and Hamiltonian are represented by

$$\Pi_i^a \rightarrow -ig^2 \frac{\delta}{\delta A_i^a}; \quad H = \int d^3x [-\frac{1}{2}g^2 (\frac{\delta}{\delta A_i^a})^2 + \frac{1}{2g^2} (B_i^a)^2] \quad (\text{A13})$$

where B_i^a is the chromomagnetic field strength. The generator of infinitesimal gauge transformations is $D_j^{ab}(-i\delta/\delta A_j^b)$, and this must annihilate ψ . The exponent S in ψ has the form given in Eq. (31), repeated here for convenience:

$$g^2 S = \frac{1}{2!} \int \int A_i^a \Omega_{ij} A_j^a + \frac{1}{3!} \int \int \int A_i^a A_j^b A_k^c \Omega_{ijk}^{abc} + \dots \quad (\text{A14})$$

Invariance of S under infinitesimal gauge transformations is trivial for the two-point function Ω_{ij} ; this quantity must be conserved, so that in Fourier space

$$\Omega_{ij}(k) = \Omega(k) P_{ij}(k); \quad P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}. \quad (\text{A15})$$

For the free theory $\Omega_0(k) = k$.

Gauge invariance is more complicated for higher-point functions. Annihilating ψ with the generator of gauge transformations yields a set of *ghost-free* Ward identities (these Ward identities also apply to the pinch technique [18, 23] construction of gauge-invariant Green's functions). For example, the Ward identity for the three-point function is:

$$k_{1i} \Omega_{ijk}^{abc}(k_1, k_2, k_3) = f^{abc} [\Omega_{jk}(2) - \Omega_{jk}(3)] \quad (\text{A16})$$

where $\Omega_{jk}(2) \equiv \Omega_{jk}(k_2)$, etc.

Now turn to the FSE itself. The equation determining the three-point function has the general form

$$\Omega_{il}(1)\Omega_{ljk}^{abc} + \Omega_{jl}(2)\Omega_{lik}^{bac} + \Omega_{kl}(3)\Omega_{lij}^{cab} = f^{abc}\Gamma_{ijk}. \quad (\text{A17})$$

The right-hand side Γ_{ijk} comes from the cubic term in H , plus another term from the five-point function. The Ward identity for Γ_{ijk} is determined by the above equation plus the Ward identities for the two- and three-point functions as already given, and multiplying both sides of Eq. (A17) by k_{1i} yields:

$$k_{1i}\Gamma_{ijk} = \Omega_{jk}^2(3) - \Omega_{jk}^2(2). \quad (\text{A18})$$

For free particles, with $\Omega = \Omega_0$, this is satisfied by the usual free three-point vertex

$$\Gamma_{ijk}^0 = i(k_1 - k_2)_k \delta_{ij} + c.p. \quad (\text{A19})$$

The reader can verify that the FSE equation (A17) has a solution of the form:

$$\Omega_{ijk}^{abc}(k_1, k_2, k_3) = [\Omega(1) + \Omega(2) + \Omega(3)]^{-1} f^{abc} \left\{ \Gamma_{ijk} + \left\{ \Omega(1) \frac{k_{1i}}{k_1^2} [\Omega_{jk}(2) - \Omega_{jk}(3)] + c.p. \right\} \right\} \quad (\text{A20})$$

which respects the Ward identity of Eq. (A16), by virtue of the massless pole terms of Eq. (A20). It should now be clear that these longitudinally-coupled massless excitations will occur, as a result of enforcing gauge invariance, for every n-point function. We will shortly identify these with couplings of the GNLS field introduced in our conjecture for the infrared-effective action.

So far the vertex function Γ_{ijk} is undetermined. As [8] argues, one can carry out a program of expressing all higher-point functions in terms of the two-point function, and then the FSE (or the equivalent dressed-loop expansion) becomes a non-linear, non-perturbative equation for this two-point function Ω . The idea, known also as the gauge technique, is to find an infrared-effective approximation to Γ_{ijk} that exactly satisfies the Ward identity (A18) for any Ω . One can, at least in principle, find such infrared-effective approximations for four- and higher-point functions as functionals of Ω . In fact, a very general form for the “solution” to the Ward identity for the three- and four-point functions is known [24, 41] for arbitrary dependence of Ω on momentum. The word “solution” is enclosed in quotes because it is not unique; any completely-conserved term can be added to the “solution” for Γ_{ijk} , for example. But the point is that purely-conserved terms are of higher order in momenta than the terms saved in the gauge technique.

The general solution of [24] is:

$$\Gamma_{ijk} = \delta_{ij}(k_1 - k_2)_k - \frac{k_{1i}k_{2j}}{2k_1^2k_2^2}(k_1 - k_2)_l \Pi_{lk}(k_3) - [P_{il}(k_1)\Pi_{lj}(k_2) - P_{jl}(k_2)\Pi_{li}(k_1)] \frac{k_{3k}}{k_3^2} + c.p. \quad (\text{A21})$$

where the first term on the right is the free vertex Γ_{ijk}^0 and $\Pi_{ij}(k) \equiv P_{ij}(k)\Pi(k)$ is the transverse pinch-technique [18, 23] self-energy, related to Ω_{ij} by:

$$\Omega_{ij}^2 = P_{ij}[\Omega_0^2 + \Pi\{\Omega\}] \quad (\text{A22})$$

where $\Omega_0^2 = k^2$ is the free gluon contribution.

In the simple case studied by us, $\Pi = M^2$, and the resulting expression for Γ_{ijk} is:

$$\Gamma_{ijk} = \delta_{ij}(k_1 - k_2)_k + \frac{M^2}{2} \frac{k_{1i}k_{2j}(k_1 - k_2)_k}{k_1^2k_2^2} + c.p. \quad (\text{A23})$$

As we saw above, in ordinary quantum mechanics in one spatial dimension x the exponent $S(x)$ of ψ can be determined systematically from a set of non-linear algebraic equations, such that each term of $\mathcal{O}(x^3)$ or higher can be expressed in terms of the quadratic coefficient Ω . Finally, Ω is determined by a single non-linear equation, equivalent to a dressed-loop expansion. Combining the pinch technique and the gauge technique gives a completely analogous program for gauge theories, based on “solving” the Ward identities insuring gauge invariance. While this program can only be carried out approximately, it is gauge-invariant by construction. Ultimately it yields a dressed-loop equation for a single transverse operator $\Omega_{ij}(k) \equiv P_{ij}(k)\Omega(k)$. The pinch-technique self-energy Π is itself a complicated function of Ω , found by using dressed propagators of the general form given in Eq. (A10), with Ω_0 replaced by Ω and with appropriate vector kinematics. In effect, Π is the on-shell self-energy and any $\pm i\Omega(k)$ occurring in Π is a fourth component of a Euclidean four-vector (k_4, k) that is on-shell, by which we mean that $k_4^2 + k^2 + M^2 = 0$, or $k_4 = \pm i\sqrt{k^2 + M^2}$.

All that we need from this development in the main text is Eq. (A23), which will be used in the large- M expansion of the three-point function Ω_{ijk}^{abc} .

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